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Invariant surfaces of a three-dimensional manifold with constant Gauss curvature

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Abstract

We give a reduction procedure to determine (locally) the surfaces with constant Gauss curvature in a three-dimensional manifold which are invariant under the action of a one-parameter subgroup of the isometry group of the ambient space. We apply this procedure to describe the invariant surfaces with constant Gauss curvature in $\mathbb{H}^2 \times \mathbb{R}$ and in $\mathbb{H}_3$.

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1. Introduction

Let (N^3, g) be a three-dimensional Riemannian manifold and let X be a Killing vector field on N. Then X generates a one-parameter subgroup G_X of the group of isometries of (N^3, g). For $x \in N$, the isotropy subgroup G_x of G is compact and the quotient space G_X/G_x

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is diffeomorphic to the orbit $G(x) = \{gx \in N : g \in G\}$. An orbit $G(x)$ is called principal if there exists an open neighbourhood $U \subset N$ of x such that for all orbits $G(y)$, $y \in U$, the isotropy subgroups G_y are conjugate. If N/G is connected, from the Principal Orbit Theorem [6], the principal orbits are all diffeomorphic and the set N_r, consisting of points belonging to principal orbits, is open and dense in N. Moreover, the quotient space N_r/G_X is a connected differentiable manifold and the quotient map $\pi : N_r \to N_r/G_X$ is a submersion.

Let now $f : M^2 \to (N^3, g)$ be an immersion from a surface M^2 into N^3 and assume that $f(M) \subset N_r$. We say that f is a G_X-*equivariant* immersion, and $f(M)$ a G_X-*invariant* surface of N, if there exists an action of G_X on M^2 such that for any $x \in M^2$ and $g \in G_X$ we have $f(gx) = gf(x)$.

A G_X-equivariant immersion $f : M^2 \to (N^3, g)$ induces on M^2 a Riemannian metric, the pull-back metric, denoted by g_f and called the G_X-*invariant induced metric*.

The aim of this paper is to give a local description of the G_X-equivariant immersions from a surface into a 3-manifold that induce on M^2 a metric of constant Gauss curvature.

2. Equivariant geometry of invariant surfaces

Let $f : M^2 \to (N^3, g)$ be a G_X-equivariant immersion from a surface M^2 into a Riemannian manifold (N^3, g) and let endow M^2 with the G_X-invariant induced metric g_f. Assume that $f(M^2) \subset N_r$ and that N/G_X is connected. Then f induces an immersion $\tilde{f} : M/G_X \to N_r/G_X$ between the orbit spaces and the space N_r/G_X can be equipped with a Riemannian metric, the *quotient metric*, so that the quotient map $\pi : N_r \to N_r/G_X$ is a Riemannian submersion.

Following [3] we shall describe the quotient metric of the regular part of the orbit space N/G_X. It is well known (see, for example [4]) that N_r/G_X can be locally parametrized by the invariant functions of the Killing vector field X. If $\{f_1, f_2\}$ is a complete set of invariant functions on a G_X-invariant subset of N_r, then the quotient metric is given by $\tilde{g} = \sum_{i,j=1}^{2} h^{ij}\, \mathrm{d}f_i \otimes \mathrm{d}f_j$ where (h^{ij}) is the inverse of the matrix (h_{ij}) with entries $h_{ij} = g(\nabla f_i, \nabla f_j)$.

We can picture the above construction by the following diagram:

$$\begin{array}{ccc} (M^2, g_f) & \xrightarrow{\ f\ } & (N^3, g) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M^2/G_X & \xrightarrow{\ \tilde{f}\ } & (N_r^3/G_X, \tilde{g}) \end{array}$$

where π is a Riemannian submersion.

We now give a local description of the G_X-invariant surfaces of N^3. Let $\tilde{\gamma} : (a, b) \subset \mathbb{R} \to (N_r^3/G_X, \tilde{g})$ be a curve parametrized by arc length and let $\gamma : (a, b) \subset \mathbb{R} \to N^3$ be a lift of $\tilde{\gamma}$, such that $\mathrm{d}\pi(\gamma') = \tilde{\gamma}'$. If we denote by ϕ_r, $r \in (-\epsilon, \epsilon)$, the local flow of the Killing vector field X, then the map

$$\psi : (a, b) \times (-\epsilon, \epsilon) \to N^3, \quad \psi(t, r) = \phi_r(\gamma(t)), \tag{1}$$

defines a parametrized G_X-invariant surface.

Conversely, if $f(M^2)$ is a G_X-invariant immersed surface in N^3, then $\tilde{f}$ defines a curve in $(N_r^3/G_X, \tilde{g})$ that can be locally parametrized by arc length. The curve $\tilde{\gamma}$ is generally called the *profile* curve.

In the sequel we need the following theorem.

Theorem 2.1 (Caddeo et al. [2]). *Let $g = E\,\mathrm{d}t^2 + 2F\,\mathrm{d}t\,\mathrm{d}r + G\,\mathrm{d}r^2$ be a Riemannian metric on a local chart (t, r) of a surface M^2 with coefficients E, F and G that depend only on t.*

(i) *If (M^2, g) has constant Gauss curvature, then there exists a constant $A \in \mathbb{R}$ such that*

$$\frac{(\mathrm{d}G/\mathrm{d}t)^2}{EG - F^2} + 4KG = A. \tag{2}$$

(ii) *Vice versa, suppose that* Eq. (2) *holds with K and A real constants. Then, in all points where $G' \neq 0$, the surface M^2 has constant Gauss curvature equal to K.*

We are now ready to state the main result.

Theorem 2.2. *Let $f : M^2 \to (N_r^3, g)$ be a G_X-equivariant immersion, $\tilde{\gamma} : (a, b) \subset \mathbb{R} \to (N_r^3/G_X, \tilde{g})$ a parametrization by arc length of $\tilde{f}$ and γ a lift of $\tilde{\gamma}$.*

(i) *If the G_X-invariant induced metric g_f is of constant Gaussian curvature K, then the function $\omega(t) = \|X(\gamma(t))\|_g$ satisfies the following differential equation*:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\omega(t) + K\omega(t) = 0. \tag{3}$$

(ii) *Vice versa, suppose that* Eq. (3) *holds with K a real constant. Then, in all points where* $\mathrm{d}(\omega^2)/\mathrm{d}t \neq 0$, *the G_X-invariant induced metric g_f has constant Gauss curvature.*

Proof. Locally the surface $f(M^2)$ can be parametrized, using (1), by $\psi(t, r) = \phi_r(\gamma(t))$. *Thus the pull-back metric can be written as* $g_f = E\,\mathrm{d}t^2 + 2F\,\mathrm{d}t\,\mathrm{d}r + G\,\mathrm{d}r^2$ where

$$E = g(\psi_t, \psi_t) = g(\mathrm{d}\phi(\gamma'), \mathrm{d}\phi(\gamma')), \qquad F = g(\psi_t, \psi_r) = g(\mathrm{d}\phi(\gamma'), X),$$
$$G = g(\psi_r, \psi_r) = g(X, X) = \omega^2.$$

Since the r-coordinate curves are the orbits of the action of the one-parameter group of isometries G_X, the coefficients of the metric do not depend on r.

Now, assume that (M^2, g_f) has constant Gauss curvature K, then, from Theorem 2.1, there exists a constant $A \in \mathbb{R}$ such that

$$\frac{(\mathrm{d}G/\mathrm{d}t)^2}{EG - F^2} + 4KG = A. \tag{4}$$

Since γ is the lift of $\tilde{\gamma}$ with respect to the Riemannian submersion π, we have that $\mathrm{d}\pi(\psi_t) = \tilde{\gamma}'$ and $\mathrm{d}\pi(\psi_r) = 0$. Let e be a local unit vector field tangent to the surface and horizontal with respect to π. Then, since $\mathrm{d}\pi(\psi_t) = \tilde{\gamma}'$ has norm 1 and π is a Riemannian submersion, ψ_t can be decomposed as

$$\psi_t = g(\psi_t, \psi_r)\frac{\psi_r}{g(\psi_r, \psi_r)} + e = \frac{F}{G}\psi_r + e.$$

Calculating the norm yields to

$$EG - F^2 = G.$$

Then Eq. (4) can be rewritten as

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)^2 + 4KG^2 = AG,$$

which is equivalent, taking into account that $G = \omega^2$, to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\omega(t) + K\omega(t) = 0.$$

Conversely, assume that along a curve γ the function $\omega(t) = \|X(\gamma(t))\|_g$ satisfies the differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\omega(t) + K\omega(t) = 0,$$

for some real constant K. Then there exist a constant A such that the function $G = \omega^2$ satisfies the differential equation:

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)^2 + 4KG^2 = AG$$

and, from Theorem 2.1, in all points where $\mathrm{d}G/\mathrm{d}t \neq 0$, the parametrization $\psi(t, r)$ defines a parametrized surface with constant Gauss curvature K. □

By integration of Eq. (3) we have the following corollary.

Corollary 2.3. *Let $f : M^2 \to (N^3, g)$ be a G_X-equivariant immersion which induces a G_X-invariant metric g_f on M^2 of constant Gauss curvature K. Then the norm $\omega(t)$ of the Killing vector field X along a lift of the profile curve is*:

- *for $K = 0$ given by*

$$\omega(t) = c_1 t + c_2, \quad c_1, c_2 \in \mathbb{R};$$

- *for* $K = 1/R^2 > 0$ *given by*

$$\omega(t) = c_1 \cos\left(\frac{t}{R}\right) + c_2 \sin\left(\frac{t}{R}\right), \quad c_1, c_2 \in \mathbb{R};$$

- *for* $K = -1/R^2 < 0$ *given by*

$$\omega(t) = c_1 \cosh\left(\frac{t}{R}\right) + c_2 \sinh\left(\frac{t}{R}\right), \quad c_1, c_2 \in \mathbb{R}.$$

As we shall show in the next section the profile curve of a G_X-invariant surface can be parametrized as a function of ω. Thus, using Corollary 2.3, we can give the explicit parametrization of the profile curve.

Remark 2.4. If $(N^3, g) = (\mathbb{R}^3, \text{can})$ is the Euclidean three-dimensional space, then the Killing vector fields generate either translations or rotations. In the case of translations the quotient space $\mathbb{R}^3/G_X$ is $\mathbb{R}^2$ with the flat metric and ω is constant. Thus, from Eq. (3), we see that any curve in the quotient space generates a flat right cylinder. In the case of rotations we can assume, without loss of generality, that the rotation is about a coordinate axis, say x_3. Then the Killing vector field is $X = -x_2(\partial/\partial x_1) + x_1(\partial/\partial x_2)$ and the quotient space is $\mathbb{R}^3/G_X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0, x_1 \geq 0\}$ with the flat metric. If $\tilde{\gamma}(t) = (u(t), 0, v(t)) \in \mathbb{R}^3/G_X$ is a arc length parametrized profile of a G_X-invariant surface, then the norm of X restricted to the profile curve is $\omega = u(t)$ and, using Corollary 2.3, we find the classical explicit parametrization of surfaces of revolution with constant Gauss curvature (see, for example [1]).

3. Invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this section we use Theorem 2.2 to describe (locally) all G-invariant surfaces with constant Gauss curvature of the product $\mathbb{H}^2 \times \mathbb{R}$ where G is a one-parameter subgroup of the isometries. Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the half plane model of the hyperbolic plane and consider $\mathbb{H}^2 \times \mathbb{R}$ endowed with the product metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + \mathrm{d}z^2.$$

Proposition 3.1. *The Lie algebra of the infinitesimal isometries of the product* $(\mathbb{H}^2 \times \mathbb{R}, \mathrm{d}s^2)$ *admits the following bases of Killing vector fields*:

$$X_1 = \frac{x^2 - y^2}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial z}.$$

Proof. This result comes from a direct integration of the Killing equation $L_X g = 0$. □

Let denote by G_i the one-parameter subgroup of isometries generated by X_i, by G_{ij} the one-parameter subgroup of isometries generated by linear combinations of X_i and X_j and so on. We first note that if the groups G_X and G_Y generated by two Killing vector fields X and Y are conjugate (we shall write $G_X \sim G_Y$) then the respectively invariant surfaces are congruent, i.e. isometric with respect to an isometry of the ambient space. Therefore, we can reduce the study of the invariant surfaces by analyzing all the conjugate one-parameter groups of isometries. In [5] there is the complete list of the conjugate groups of isometries in $\mathbb{H}^2 \times \mathbb{R}$ which gives the following lemma.

Lemma 3.2 (Onnis [5]). *Any surface in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant under the action of a one-parameter subgroup of isometries G_X, generated by a Killing vector field $X = \sum_i a_i X_i$, is isometric to a surface invariant under the action of one of the following groups*:

$$G_{24}, \quad G_{34}, \quad G_{12}^*, \quad G_{124}^*,$$

where G_{12}^* is the one-parameter group generated by $X_{12}^* = X_1 + (X_2)/2$ and G_{124}^* is the one-parameter group generated by X_{12}^* and X_4.

Now let G be a one-parameter group of isometries among those described in Lemma 3.2 and let $\tilde{\gamma} = (u(s), v(s))$ be a curve parametrized by arc length in the orbit space $\mathcal{B} = (\mathbb{H}^2 \times \mathbb{R})/G$ endowed with the quotient metric $\tilde{g}$, that is such that

$$\tilde{g}(\tilde{\gamma}', \tilde{\gamma}') = \tilde{g}_{11} {u'}^2 + 2\tilde{g}_{12} u' v' + \tilde{g}_{22} {u'}^2 = 1,$$

where with $'$ we have denoted the derivative with respect to s.

If we denote by ω the norm of the Killing vector field X that generates the one-parameter group of isometries G, we can give the following local description of the G-invariant surfaces of $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 3.3. *Let $\tilde{\gamma} = (u(s), v(s))$ be a curve in the orbit space $(\mathbb{H}^2 \times \mathbb{R}/G, \tilde{g})$ parametrized by arc length, which is the profile curve of a G-invariant surface in $(\mathbb{H}^2 \times \mathbb{R})$. Then*:

- *If $G = G_4$, the orbit space is $\mathbb{H}^2$ and any curve parametrized by arc length is the profile curve of a flat G_4-invariant cylinder.*
- *If $G = G_{24}$, the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u > 0\}$ with the orbital metric $\tilde{g} = (\mathrm{d}u^2/u^2) + (\mathrm{d}v^2/(a^2 + b^2 u^2))$ and the profile curve can be parametrized by*

$$u(s) = |a|/\sqrt{\omega^2 - b^2}, \quad a, b \in \mathbb{R},$$

$$v(s) = \int_{s_0}^{s} \sqrt{\frac{a^2\omega^2}{\omega^2 - b^2}\left[1 - \left(\frac{\omega\omega'}{\omega^2 - b^2}\right)^2\right]}\,\mathrm{d}t.$$

  As a special case, when $a = 1$ and $b = 0$, we have the G_2-invariant surfaces.
- *If $G = G_{34}$, then the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : 0 < u < \pi\}$ with the orbital metric $\tilde{g} = (\mathrm{d}u^2/\sin^2(u)) + (\mathrm{d}v^2/(a^2 + b^2 \sin^2(u)))$ and the profile curve can be parametrized*

by

$$u(s) = \arcsin\left(|a|/\sqrt{\omega^2 - b^2}\right), \quad a, b \in \mathbb{R},$$

$$v(s) = \int_{s_0}^{s} \sqrt{\frac{a^2\omega^2}{\omega^2 - b^2}\left[1 - \frac{(\omega\omega')^2}{(\omega^2 - b^2)(\omega^2 - a^2 - b^2)}\right]}\,\mathrm{d}t.$$

As a special case, when $a = 1$ *and* $b = 0$, *we have the* G_3*-invariant surfaces.*

- *If* $G = G^*_{124}$, *then the orbit space is* $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u \geq 2\}$ *with the orbital metric* $\tilde{g} = (\mathrm{d}u^2/(u^2 - 4)) + [(u^2 - 4)\,\mathrm{d}v^2/((u^2 + 4(a^2 - 1)))]$ *and the profile curve can be parametrized by*

$$u(s) = 2\sqrt{\omega^2 + 1 - a^2}, \quad a \in \mathbb{R},$$

$$v(s) = \int_{s_0}^{s} \sqrt{\frac{\omega^2}{\omega^2 - a^2}\left[1 - \frac{(\omega\omega')^2}{(\omega^2 - a^2)(\omega^2 + 1 - a^2)}\right]}\,\mathrm{d}t.$$

As a special case, when $a = 0$, *we have the* G^*_{12}*-invariant surfaces.*

Proof. The proof is a direct consequence of the expression of the quotient metric and of ω. We shall write down the explicit calculations for the G_{24}-invariant surfaces leaving the other cases to the reader.

The Killing vector field generating the Lie algebra of G_{24} is $X = a(\partial/\partial x) + b(\partial/\partial z)$, $a, b \in \mathbb{R}$. A set of two invariant functions is

$$u = y > 0, \qquad v = bx - az.$$

Thus the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u > 0\}$ and the orbital metric is

$$\tilde{g} = \frac{\mathrm{d}u^2}{u^2} + \frac{\mathrm{d}v^2}{a^2 + b^2u^2}.$$

Let $\tilde{\gamma} = (u(s), v(s))$ be a curve parametrized by arc length in the orbit space $\mathcal{B}$. Then the square of the norm of the Killing vector filed X along a lift of $\tilde{\gamma}$, with respect to g, is

$$\omega^2 = \|X\|_g^2 = (a^2 + b^2u^2)/u^2. \tag{5}$$

Finally, from (5), we find immediately that $u(s) = |a|/\sqrt{\omega^2 - b^2}$ and, from

$$\frac{(u')^2}{u^2} + \frac{(v')^2}{a^2 + b^2u^2} = 1,$$

by integration, we find the desired expression for $v(s)$. □

Now if $\tilde{\gamma}$ is the profile curve of a G-invariant surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gauss curvature, then the explicit parametrization of $\tilde{\gamma}$ can be obtained by replacing in Theorem 3.3 the corresponding expression of the function ω, according to the value of the Gauss curvature K, as we have described in Corollary 2.3. For example, in the case of the G^*_{12}-invariant surfaces of $\mathbb{H}^2 \times \mathbb{R}$, for some values of ω we have:

(1) If $K = 0$, choosing $\omega(s) = s$, we have the following parametrization for the profile curve

$$\tilde{\gamma}(s) = (2\sqrt{s^2+1}, \sqrt{s^2+1}).$$

(2) If $K > 0$, choosing $\omega(s) = \cos s$, the corresponding profile curve is

$$\tilde{\gamma}(s) = \left(2\sqrt{\cos^2 s + 1}, \frac{\sqrt{2\cos^2 s}}{\cos s}\arctan\left(\frac{\sin}{\sqrt{\cos^2 s + 1}}\right)\right).$$

(3) If $K < 0$, taking the function $\omega(s) = \sinh s$, we obtain

$$\tilde{\gamma}(s) = (2\cosh s, 0).$$

4. Invariant surfaces in the Heisenberg group $\mathbb{H}_3$

The three-dimensional Heisenberg space $\mathbb{H}_3$ is the two-step nilpotent Lie group standardly represented in $Gl_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

with $x, y, z \in \mathbb{R}$. Endowed with the left-invariant metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \left(\frac{1}{2}y\,\mathrm{d}x - \frac{1}{2}x\,\mathrm{d}y + \mathrm{d}z\right)^2,$$

$(\mathbb{H}_3, g)$ has a rich structure, reflected by the fact that its group of isometries is of the dimension 4, which is the maximal possible for a non-constant curvature 3-manifold. In particular we have the following proposition.

Proposition 4.1. *The Lie algebra of the infinitesimal isometries of the product* $(\mathbb{H}_3, \mathrm{d}s^2)$ *admits the following bases of Killing vector fields*

$$X_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

According to [3] the one-dimensional subgroups of the isometry group $Isom(\mathbb{H}_3, \mathrm{d}s^2)$ can be divided in two families:

(1) The one-parameter subgroups generated by linear combinations

$$a_1X_1 + a_2X_2 + a_3X_3 + bX_4,$$

of Killing vector fields, with $b \neq 0$. These subgroups are called of *helicoidal type*. If $a_i = 0$ for $i \in \{1, 2, 3\}$, we obtain the group SO(2) generated by X_4.

(2) The one-parameter subgroups generated by linear combinations of X_1, X_2 and X_3, called of *translational type*.

A surface in the Heisenberg space is called *helicoidal* or *translational* if it is invariant under the action of a helicoidal or a translational one-parameter subgroup of isometries respectively.

To describe (locally) all G-invariant surfaces with constant Gauss curvature of the Heisenberg space, where G is a one-parameter subgroup of isometries, we use the following lemma.

Lemma 4.2 (Figueroa et al. [3]). *A surface in $\mathbb{H}_3$ which is invariant under the action of a one-parameter subgroup of isometries G_X generated by a Killing vector field $X = \sum_i a_i X_i$ is isometric to a surface invariant under the action of one of the following groups*:

$$G_1, \quad G_3, \quad G_{34}.$$

With the same argument as in Theorem 3.3 we can give the following local description of the G-invariant surfaces in $\mathbb{H}_3$.

Theorem 4.3. *Let $\tilde{\gamma} = (u(s), v(s))$ be a curve in the orbit space $(\mathbb{H}_3/G, \tilde{g})$ parametrized by arc length which is the profile curve of a G-invariant surface in $\mathbb{H}_3$.*

- *If $G = G_1$, then the orbit space is $\mathcal{B} = \mathbb{R}^2$ with the orbital metric $\tilde{g} = \mathrm{d}u^2 + (\mathrm{d}v^2/(u^2 + 1))$ and the profile curve can be parametrized by*

$$u(s) = \sqrt{\omega^2 - 1}, \qquad v(s) = \int_{s_0}^{s} \sqrt{\omega^2 \left[1 - \frac{(\omega\omega')^2}{(\omega^2 - 1)}\right]}\, \mathrm{d}t.$$

- *If $G = G_3$, then the orbit space is $\mathcal{B} = \mathbb{R}^2$ with the orbital metric $\tilde{g} = \mathrm{d}u^2 + \mathrm{d}v^2$ and, as $\omega = 1$, any curve parametrized by arc length is the profile curve of a flat G_3-invariant vertical cylinder.*
- *If $G = G_{34}$, then the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u \geq 0\}$ with the orbital metric $\tilde{g} = \mathrm{d}u^2 + (4u^2\, \mathrm{d}v^2/(4u^2 + (u^2 + 2a)^2))$ and the profile curve can be parametrized by*

$$u(s) = \sqrt{2\left(\sqrt{\omega^2 + 2a + 1} - a - 1\right)}, \quad a \in \mathbb{R}^+,$$

$$v(s) = \int_{s_0}^{s} \frac{\omega}{2} \sqrt{\frac{2(\omega^2 + 2a + 1)^{3/2} - (2a + 1)\omega^2 - 4a^2 - 6a - 2 - \omega'^2}{(\omega^2 + 2a + 1)(\sqrt{\omega^2 + 2a + 1} - a - 1)^2}}\, \mathrm{d}t.$$

As a special case, when $a = 0$, *we have the* SO(2)-*invariant surfaces*.

Now if $\tilde{\gamma}$ is the profile curve of a G-invariant surface in $\mathbb{H}_3$ with constant Gauss curvature, then the explicit parametrization of $\tilde{\gamma}$ can be obtained, as we have done for the invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$, by replacing in Theorem 3.3 the corresponding expression of the function ω, according to the value of the Gauss curvature K, as we have described in Corollary 2.3.

Remark 4.4. The case of SO(2)-invariant surfaces with constant Gauss curvature was described explicitly by Caddeo–Piu–Ratto in [2].

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References

[1] R. Caddeo, A. Gray, Curve e Superfici, vol. I, CUEC, 2000.
[2] R. Caddeo, P. Piu, A. Ratto, Rotational surfaces in H_3 with constant Gauss curvature, Boll. Un. Mat. Ital. B (7) (1996) 341–357.
[3] C.B. Figueroa, F. Mercuri, R.H.L. Pedrosa, Invariant surfaces of the Heisenberg groups, Ann. Mater. Pure Appl. 177 (4) (1999) 173–194.
[4] P.J. Olver, Application of Lie Groups to Differential Equations, GTM 107, Springer-Verlag, New York, 1986.
[5] I.I. Onnis, Ph.D. Thesis, University of Campinas, 2005.
[6] R.S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. Math. (73) (1961) 295–323.